# Towards resolving generic singularity problem of general relativity

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# OUTLINE

# Introduction

Evidence for the existence of cosmological singularity

- observational cosmology:
  the Universe has been expanding for nearly 14 billion years (emerged from a state with extremely high energy densities of physical fields)
- theoretical cosmology:
  almost all known general relativity models of the Universe (Lemaître, Kasner, Friedmann, Bianchi, Szekeres, ...) predict the existence of cosmological singularities (diverging gravitational and matter field invariants, incomplete geodesics)
- Hawking and Penrose theorems:
  our universe must have been singular a finite time ago (geodesic incompleteness)

Existence of singularities means that classical GR is incomplete.

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Hypothesis: the Universe was in a quantum phase a finite time ago.

Some questions concerning quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition:
  quantum phase $\rightleftarrows$ classical phase?
- How to relate quantum theory with cosmic observations?
  - What is the origin of inflation?
  - What is the structure of tiny fluctuations visible in CMB?
  - What is the spectrum of primordial gravitational waves?
- Suppose the notion of time is well defined in quantum phase:
  - How long had the quantum phase lasted?
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- Canonical quantization based on the Holst action and loop geometry
  - Dirac's approach := 'first quantize then impose constraints'
  - RPS approach := 'first solve constraints then quantize'
- Hybrid quantization: mix of coherent states and canonical methods based on the Hilbert-Einstein action

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- Cosmic singularity problem of FRW model can be resolved: classical big bang $\longrightarrow$ quantum big bounce
- Evolution of quantum phase can be described in terms of self-adjoint physical (true) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during quantum evolution of universe.

The FRW model underlies the standard model of cosmology that is successfully used to describe available data of observational cosmology.

However, we cannot be happy with classical/quantum FRW.

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- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)[1]
- Dynamics of anisotropic models like Bianchi VIII and Bianchi IX has been analyzed to get insight into the dynamics of spacetime near the singularity[2]
- BKL scenario/conjecture is generic 'solution' to GR near CS [3]
  - corresponds to non-zero measure subset of all initial conditions
  - is stable against perturbation of initial conditions
- support for BKL from numerical simulations of the approach to singularity[4]
- analytic support for BKL obtained within Hubble-normalized dynamical system aproach[5]

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- BKL in string theory[6]
  - appears in the low energy limits of bosonic sectors of all five types of superstring models
  - Lorenzian hyperbolic Kac-Moody algebra underlies asymptotic structure of spacetime near cosmological singularity

Big challenge: quantization of BKL scenario.

Application of non-singular quantum BKL

- realistic model of the very early Universe
- model resolving the singularity problem of black holes
- may help in construction of theory unifying gravitation and quantum physics.

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# The Bianchi IX model:

- Dynamics of Bianchi IX, near the singularity, is the best prototype for the BKL scenario[7]
- Questions to be answered:
  - What happens to the classical singularity of BIX at the quantum level?
  - What happens to the chaotic dynamics of BIX at the quantum level?
  - What is the generation of primordial GW for classical/quantum BIX?
- Successful quantization of the Bianchi IX model would open door to the quantization of the BKL scenario.

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# Metric of the Bianchi IX model

The general form of a line element of the Bianchi IX model, in the synchronous reference system, reads:

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \qquad (1)$$

where $a, b, \ldots$ run from 1 to 3 and label frame vectors; $\alpha, \beta, \ldots$ take values $1, 2, 3$ and concern space coordinates, and where $\gamma_{ab}$ is a spatial metric.

The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model.

The cosmological time variable $t$ is redefined as follows:

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where $\gamma$ is the volume density, and $\gamma \to 0$ denotes the singularity.

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# Equations of motion

Near the cosmological singularity one can assume[8]

- the stress-energy tensor components can be ignored
- the Ricci tensor components $R_a^0$ have negligible influence on the dynamics
- the anisotropy of space may grow without bound

which leads to enormous simplification of the mathematical form of the dynamics.

[8] V. A. Belinskii, I. M. Khalatnikov and M. P. Ryan, “The oscillatory regime near the singularity in Bianchi-type IX universes”, Preprint order **469** (1971), Landau Institute for Theoretical Physics, Moscow (unpublished); published as sections 1 and 2 in: M. P. Ryan, Ann. Phys. **70** (1971) 301; V. A. Belinski, “On the cosmological singularity,” Int. J. Mod. Phys. D **23**, 1430016 (2014)

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# Equations of motion (cont)

The asymptotic form (near the cosmological singularity) of the dynamical equations of general Bianchi IX model reads:

$$\frac{d^2 \ln a}{d\tau^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{d\tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{d\tau^2} = a^2 - \frac{c}{b}, \tag{3}$$

where $a = a(\tau),\ b = b(\tau),\ c = c(\tau)$ are directional scale factors.

The solutions to (3) must satisfy the constraint:

$$\frac{d \ln a}{d\tau}\frac{d \ln b}{d\tau} + \frac{d \ln a}{d\tau}\frac{d \ln c}{d\tau} + \frac{d \ln b}{d\tau}\frac{d \ln c}{d\tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{4}$$

# State of asymptotic silence[9]

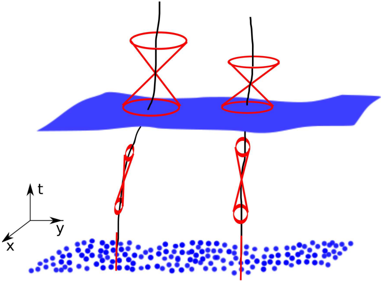


Figure: Collapse of the light cones while approaching the state of asymptotic silence.

Asymptotic silence (locality conjecture)

- characterized by causal disconnection of space points at large curvature of spacetime (sector of classical BIX)
- expected to have quantum counterpart in quantum gravity at large energy densities (sector of quantum BIX)

[9]figure done by J Mielczarek

# Lagrangian and Hamiltonian

Eq (3) can be obtained from the Lagrangian equations of motion with $L$ in the form:

$$L := \dot{x}_1\dot{x}_2 + \dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_3 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2). \quad (5)$$

The momenta, $p_I := \partial L/\partial \dot{x}_I$, are:

$$p_1 = \dot{x}_2 + \dot{x}_3, \quad p_2 = \dot{x}_1 + \dot{x}_3, \quad p_3 = \dot{x}_1 + \dot{x}_2. \quad (6)$$

The Hamiltonian of the system:

$$H := p_I\dot{x}_I - L = \frac{1}{2}(p_1p_2 + p_1p_3 + p_2p_3) \quad (7)$$

$$-\frac{1}{4}(p_1^2 + p_2^2 + p_3^2) - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2),$$

which due to (6) and (4) leads to the dynamical constraint:

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# Hamilton's equations

The Hamilton equations have the following explicit form:

$$\dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \tag{9}$$

$$\dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \tag{10}$$

$$\dot{x}_3 = \frac{1}{2}(p_1 + p_2 - p_3), \tag{11}$$

$$\dot{p}_1 = 2\exp(2x_1) - \exp(x_2 - x_1), \tag{12}$$

$$\dot{p}_2 = \exp(x_2 - x_1) - \exp(x_3 - x_2), \tag{13}$$

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# Dynamical systems method

- The local geometry of the phase space is characterized by the nature and position of its critical points. These points are locations where the derivatives of all the dynamical variables vanish.
- The set of all critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, may provide a qualitative description of a given dynamical system.
- The above situation is specific to the case when a fixed point is of the hyperbolic type. In the case of the nonhyperbolic fixed point, linearized vector field at the fixed point cannot be used to specify local properties of the phase space. Nearby points may have completely different neighborhood of orbits.

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# Dynamical systems analysis (cont)

The set of critical points $S_B$ is found to be:

$$\begin{aligned} S_B : \quad = \quad & \{(x_1, x_2, x_3, p_1, p_2, p_3) \in \bar{\mathbb{R}}^6 \mid (x_1 \to -\infty,\ x_2 \to -\infty,\ x_3 \to -\infty) \\ & \wedge (x_3 < x_2 < x_1 < 0);\ \ p_1 = 0 = p_2 = p_3\}, \end{aligned} \tag{16}$$

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

The characteristic polynomial associated with Jacobian $J$ is: $P(\lambda) = \lambda^6$, so the eigenvalues are the following: $(0, 0, 0, 0, 0, 0)$ .

Since real parts of all eigenvalues of the Jacobian are equal to zero, the set $S_B$ consists of nonhyperbolic critical points.

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Summary:

- We are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field.
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Intriguing question[10]:
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# Hamiltonian structure on physical phase space[11]

We turn our system (9)-(14) with Hamiltonian constraint, $H = 0$, into a new dynamical system with Hamiltonian to be a generator of an evolution. We call it a physical (true) Hamiltonian. To achieve that we should transform canonically the symplectic 2-form $\omega := \sum_{k=1}^{3} \left( dx_k \wedge dp_k \right)$ of kinematical phase space into canonical 2-form, $\Omega := \omega_{|_{H=0}}$, defined in physical phase space.
The Hamiltonian structure in the physical phase space is defined by the factorization:

$$\Omega = \sum_{\alpha=1}^{2} \left( dq_\alpha \wedge d\pi_\alpha \right) + dT \wedge dH_T, \tag{17}$$

where $q_\alpha, \pi_\alpha$ and $T$ are new canonical variables, and where $H_T = H_T(q_\alpha, \pi_\alpha, T)$ to be determined from the Hamiltonian constraint $H = 0$.

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# Semi-classical Bianchi IX model

In the Misner like variables the dynamics of gravitational field can be described as motion of a massless particle in 3-dimensional Minkowskian space in a potential dependent on space and time.

The Hamiltonian (constraint) for the vacuum BIX reads[12]

$$\mathrm{H} = \frac{9}{4} p^2 + 36 n^2 q^{2/3} - \mathrm{H}_q \approx 0 \,, \tag{24}$$

where $\mathrm{H}_q$ is the $q$-dependent Hamiltonian for the anisotropic variables,

$$\mathrm{H}_q := \frac{p_+^2 + p_-^2}{q^2} + 36 q^{2/3} V_n(\beta) \,. \tag{25}$$

where $(q, p; \beta\pm, p_\pm)$ are canonical variables, and where $n = 1$ or $n^3 = 16\pi^2$.

The closed FRW model can be obtained by taking $p_\pm = 0 = \beta_\pm$, or simply $\mathrm{H}_q = 0$.

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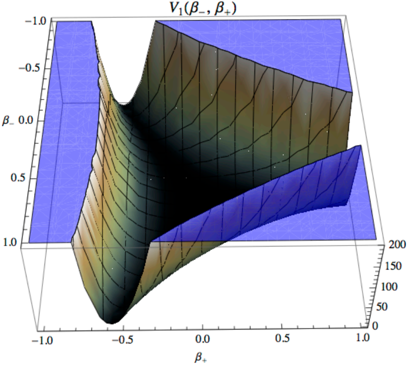


Figure: The plot of $V_n$ for $n = 1$ near its minimum.
Boundedness from below, confining aspects, three canyons, and $C_{3v}$ symmetry are illustrated.

# Classical Hamiltonian

One can rewrite the Hamiltonian in the form:

$$\mathrm{H} = \frac{9}{4} p^2 - \frac{p_+^2 + p_-^2}{q^2} - 36 q^{2/3} W_n(\beta_\pm), \tag{26}$$

where $W_n(\beta) = -n^2 + V_n(\beta)$

It results from Eq. (26) that near the singularity, $q = 0$, we may treat $q$ as heavy degree of freedom (as 'mass' of $q$ is fixed), and $\beta_\pm$ as light degrees of freedom (as 'mass' of the $\beta_\pm$ behaves as $q^2$).
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# Quantum Hamiltonian

In what follows we apply the modified Dirac quantization method:

- quantizing H in kinematical phase space
- finding the semi-classical expression $\check{\mathrm{H}}$ of the quantum Hamiltonian $\hat{\mathrm{H}}$ using the adiabatic approximation
- implementing the Hamiltonian constraint on the semi-classical level $\check{\mathrm{H}} = 0$

Since $(q, p) \in \mathbb{R}_+^* \times \mathbb{R}$ and $(\beta_\pm, p_\pm) \in \mathbb{R}^4$, we apply:

- affine coherent states quantization to $(q, p)$, which gives $\hat{p} = -i\hbar\partial_x$ and the multiplication operator $\hat{q}$, both acting in the Hilbert space $L^2(\mathbb{R}_+^*, dx)$
- canonical quantization to $(\beta_\pm, p_\pm)$, which yields $\hat{p}_\pm = -i\hbar\partial_{\beta_\pm}$ and the multiplication operator $\hat{\beta}_\pm$, both acting in $L^2(\mathbb{R}^2, d\beta_+ d\beta_-)$

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# ACS quantization: compendium[13]

Isotropy sector of phase space: $\Pi_+ := \{(q,p) \,|\, p \in \mathbb{R}\,,\ q > 0\}$
$\Pi_+$ is an affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line with multiplication:

$$(q,p)(q_0,p_0) = (qq_0, p_0/q + p),\ q \in \mathbb{R}_+^*,\ p \in \mathbb{R}\,. \tag{27}$$

$\mathrm{Aff}_+(\mathbb{R})$ has UIR realized in $\mathcal{H} = L^2(\mathbb{R}_+^*, \mathrm{d}x)$:

$$U(q,p)\phi(x) = (e^{ipx}/\sqrt{q})\phi(x/q)\,. \tag{28}$$

All affine coherent states are defined as: $|q,p\rangle = U(q,p)|\psi\rangle$, where $|\psi\rangle \in L^2(\mathbb{R}_+^*, \mathrm{d}x) \cap L^2(\mathbb{R}_+^*, \mathrm{d}x/x)$, called "fiducial vector"
Quantization of classical observable $f(q,p)$ reads:

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We note in (30) the repulsive centrifugal potential term $\hbar^2 \mathfrak{K}_1 \hat{q}^{-2}$. It results from the ACS quantization. As the universe approaches the singularity, $q \to 0$, this centrifugal term sharply grows in dynamical significance, and it is responsible for the resolution of the singularity.

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# Born-Oppenheimer approximation

In this approximation we assume that the anisotropy degrees of freedom are frozen in some eigenstate $|\phi_n(q(t)) >$, evolving adiabatically, of the $q$-dependent Hamiltonian $\hat{\mathrm{H}}_q$.

If we denote by $E_N(q)$ the eigenenergies of $\hat{\mathrm{H}}_q$, the reduced Hamiltonian $\hat{\mathrm{H}}_N^{\mathrm{red}}$ reads

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where in the harmonic approximation $E_N(q)$ $(N = 0, 1, \dots)$ are

$$E_N(q) \simeq \frac{24\hbar}{q^{2/3}} n \sqrt{2\mathfrak{K}_2\mathfrak{K}_3}\,(N+1)\,. \tag{33}$$

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Finally, we obtain

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# Imposition of Hamiltonian constraint:

Rewritten in terms of the scale factor $a^6 := q_1 q_2 q_3$ (where $q_k$ are diagonal elements of the metric), the constraint $\check{\mathrm{H}}_N^{\mathrm{red}} = 0$ reads

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- anisotropy degrees of freedom produce radiation-like energy density $\rho(a)$
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# Semiclassical trajectories (resolution of singularity)

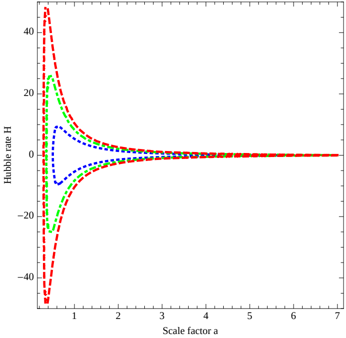


Figure: Three periodic semiclassical trajectories in the half-plane $(a, H)$. Blue dotted curve for $N = 0$, green dotdashed for $N = 1$ and red dashed for $N = 2$.

Each periodic trajectory includes quantum bounce and classical recollapse. $N = 0, 1, 3, \dots$ label discrete eigenenergies $E_N$ of anisotropic part of Hamiltonian.

# Summary

Applying

- mixed procedure of quantization (ACS and canonical)
- adiabatic approximation to the quantum Hamiltonian
- imposition of Hamiltonian constraint at the semi-classical level,

it is possible to find a quantum version of the Bianchi IX model.

Origin of repulsive force giving singularity avoidance (FRW case):

- LQC: results from approximating the curvature of connection by holonomies around small loops with non-zero size; keeping this size to be non-vanishing prevents volume density from collapsing to zero
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# Next steps for the Bianchi IX model:

- Quantization of diagonal BIX model by using the vibronic approximation: sensitivity to crossing of different energy levels which enables examination of quantum chaos (suppressed in adiabatic approximation).
- Quantization of general BIX model by using hybrid method within RPS approach
  - quantization of physical Hamiltonian to get generator of dynamics
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