Universität Bielefeld

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Determinantal structure of eigenvector correlations in the complex Ginibre ensemble

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RMT: Applications in the Information Era

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with R. Tribe, A. Tsareas & O. Zaboronski [arXiv: 1809:05905]

Plan of the talk

I) Introduction: Main questions and motivation

II) Review of Chalker and Mehlig

III) Eigenvector correlations as determinantal point processes

IV) Large-N limits and link to density correlations

V) Summary and open questions

Setup: Eigenvectors in the Ginibre ensemble

- complex Ginibre ensemble [Ginibre, 1965]
  $J_{ij} \in \mathcal{N}_{\mathbb{C}}(0, 1)$: $\langle J_{ij}\, J^*_{kl} \rangle = \delta_{ik}\delta_{jl}$, all other zero

  $N \times N$ independent complex Gaussian matrix elements
- distribution of all matrix elements $\boxed{\mathcal{P}(J) \sim \exp\left[-\mathrm{Tr}\, JJ^\dagger\right]}$

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- Correlations of λ_α well understood - eigenvectors?

Motivation: standard random matrix paradigm

- strongly coupled/ many-body Hamiltonian $\mathcal{H}$

  $\rightarrow \exists$ spectral aspects described by random matrix H of same symmetry [BGS vs. BT conjecture], e.g.
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- <u>role of eigenvectors:</u>

  - sensitive to perturbations $\rightarrow$ [talk by W. Tarnowski]
  - time dependent Brownian motion in Ginibre: [Burda et al. '14]
  $\rightarrow$ coupled evolution of $\lambda_\alpha, L_\alpha, R_\alpha$ ($\neq$ GUE or normal J)

Approach of Chalker & Mehlig

- consider expectation values $\langle \mathcal{O} \rangle := \int [dJ] \mathcal{O}(J) \mathcal{P}(J)$ of the **overlapp matrix** $\boxed{\mathcal{O}_{\alpha\beta} = (L_\alpha, L_\beta)(R_\alpha, R_\beta)}$

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$$\boxed{D_{11}(z) := \langle \sum_{\alpha=1}^{N} \mathcal{O}_{\alpha\alpha} \delta(z - \lambda_\alpha) \rangle = N \langle \mathcal{O}_{11} \delta(z - \lambda_1) \rangle}$$

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$$\boxed{D_{12}(z_1, z_2) := \langle \textstyle\sum_{\alpha \neq \beta = 1}^{N} \mathcal{O}_{\alpha\beta}\, \delta(z - \lambda_\alpha) \delta(z - \lambda_\beta) \rangle}$$
$$= N(N-1) \langle \mathcal{O}_{12}\, \delta(z - \lambda_1) \delta(z - \lambda_2) \rangle$$

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- can be expressed in terms of integrals over λ_α only [Chalker, Mehlig '98, '99]

Methods and alternative quantities

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$\boxed{\langle |\mathcal{O}_{12}|^2\rangle}$ and $\boxed{\langle \mathcal{O}_{11}\mathcal{O}_{22}\rangle}$ [Bourgade, Dubach '18]

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- **real and quaternionic Ginibre** eigenvector correlations known, cf. [Dubach '18; Förster '18] & products [Burda, Spisak, Vivo '16]

Joint densities

- Schur decomposition $\boxed{J = U(\Lambda + T)U^\dagger}$
  where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ complex eigenvalues
  T complex upper triangular, $U \in U(N)$
- $\mathrm{tr}\, JJ^\dagger = \mathrm{Tr}\,(\Lambda\Lambda^\dagger + TT^\dagger)$

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$$\boxed{P(J) \to P(\Lambda, T, U) \sim \exp\left[-\sum_{\alpha=1}^{N} |\lambda_\alpha|^2 - \sum_{k<l} |T_{kl}|^2\right] |\Delta_N(\Lambda)|^2}$$

with **Vandermonde determinant**

$$\boxed{\Delta_N(\Lambda) = \prod_{j>k}^{N}(\lambda_j - \lambda_k) = \begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{N-1} \\ 1 & \lambda_2 & & \ldots \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}}$$

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- eigenvectors depend on $T \Rightarrow$ average over T nontrivial

  $\langle \mathcal{O} \rangle_T := \int [dT] \mathcal{O}(\Lambda, T) P(T)$

Performing the T-average

<u>making T-dependence explicit:</u> $J \to \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$

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- averages over Λ and T factorise [Chalker, Mehlig '99]:

$$\boxed{\langle \mathcal{O}_{11} \rangle_T = \prod_{k=2}^{N} \left(1 + \frac{1}{|\lambda_1 - \lambda_k|^2}\right)}$$

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- holds whenever T-average remains Gauß, e.g. in

  - induced or elliptic Ginibre, and quasi-harmonic potentials

Diagonal overlap as complex eigenvalue average

$$
\begin{aligned}
D_{11}(z_1) &= N\langle \mathcal{O}_{11}\delta(z_1-\lambda_1)\rangle \\
&= \frac{N\, e^{-|z_1|^2}}{Z_N}\int d^2\lambda_2\cdots d^2\lambda_N |\Delta_N(z_1,\lambda_2,\ldots)|^2 \\
&\quad \times \prod_{l=2}^{N} e^{-|\lambda_l|^2}\left(1+\frac{1}{|z_1-\lambda_l|^2}\right)
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&\quad \times \prod_{l=2}^{N} e^{-|\lambda_l|^2}\left(|z_1-\lambda_l|^2+1\right)
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- same structure as complex eigenvalue correlations, with new weight $w_{11}(\lambda_l) = e^{-|\lambda_l|^2}\left(1+|z_1-\lambda_l|^2\right)$

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- same structure as complex eigenvalue correlations, with new weight $w_{11}(\lambda_l) = e^{-|\lambda_l|^2}\left(1+|z_1-\lambda_l|^2\right)$
- define D_{11} with more constraints
  $D_{11}(z_1,z_2) = (N-1)$ above $\times\ \delta(z_2-\lambda_2)$ etc.

Off-diagonal from diagonal overlap

$$
\begin{aligned}
D_{12}(z_1, z_2) &= \frac{-N(N-1)e^{-|z_1|^2-|z_2|^2}}{Z_N} \int d^2\lambda_3 \cdots d^2\lambda_N |\Delta_{N-3}(\lambda_3, \ldots)|^2 \\
&\times \prod_{l=3}^{N} e^{-|\lambda_l|^2} (z_1^* - \lambda_l^*)(z_2 - \lambda_l)\Big((z_1 - \lambda_l)(z_2^* - \lambda_l^*) + 1\Big)
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&\times (|z_1 - z_2|^2 + 1) \prod_{l=3}^{N} e^{-|\lambda_l|^2} |z_2 - \lambda_l|^2 \Big(|z_1 - \lambda_l|^2 + 1 \Big)
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$$\times (|z_1-z_2|^2+1)\prod_{l=3}^{N} e^{-|\lambda_l|^2}|z_2-\lambda_l|^2\Big(|z_1-\lambda_l|^2+1\Big)$$

- **Lemma1**[ATTZ '19] Define $\hat{\Pi}f(z_1,z_1^*,z_2,z_2^*) = f(z_1,z_2^*,z_2,z_1^*)$

$$\Rightarrow \boxed{D_{12}(z_1,z_2) = \frac{-e^{-|z_1-z_2|^2}}{1-|z_1-z_2|^2}\hat{\Pi}D_{11}(z_1,z_2)}$$

Reminder orthogonal polynomials on $\mathbb{C}$

- Reminder complex **eigenvalue** *k*-point corrlation functions:

$$\rho(\lambda_1, \ldots, \lambda_k) := \frac{N!}{(N-k)!Z_N} \int d^2\lambda_{k+1} \ldots d^2\lambda_N |\Delta_N(\Lambda)|^2 \prod_{j=1}^{N} w(\lambda_j)$$

$$= \det_{1\leq i,j\leq k} \left[K(\lambda_i, \lambda_j^*) \right] \quad \text{Det Point Process}$$

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- e.g. Ginibre: $w(u) = e^{-|u|^2}$, $P_l(u) = Q_l(u) = u^l, \ h_l = \pi l!$

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- almost like Ginibre, heuristic argument (translational invariance) leads to large-N result

Determinantal structiure of the conditional overlap

- **Theorem 1** [A,Tribe, Tsareas, Zaboronski '19]

$$D_{11}(\lambda_1, \ldots, \lambda_k) = \frac{1}{\pi} e^{-|\lambda_1|^2} f_{N-1}(|\lambda_1|^2) \det_{\mathbf{2} \leq i,j \leq k} \left[K_{11}(\lambda_i^*, \lambda_j) \right]$$

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- $D_{12}(\lambda_1,\dots,\lambda_k)$ can be defined similarly and follows from a generalisation of Lemma 1 to all $k>2$

Idea of the proof

- $\exists$ alternative way to express the kernel for arbitrary weight:

moment matrix $M_{ij} :=< u^i, u^j >= \int d^2\lambda(\lambda^*)^i\lambda^j$

$$\Rightarrow \boxed{K(u^*, v) = (w(u)w(v))^{\frac{1}{2}} \sum_{i,j=0}^{N-1}(u^*)^i(M^{-1})_{ij}v^j}$$

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- a very tedius calculation leads to a form that is amenable to the large-N limit

Large-N limits: Global vs. local regime

- Reminder eigenvalues density correlations:

  - global density: circular law
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- example for off-diagonal overlapp: [Chalker, Mehlig '99] heuristic

$$D_{12}^{\text{bulk}}(\lambda_1, \lambda_2) = \frac{1}{\pi^2|\lambda_1-\lambda_2|^4}\left(1-(1+|\lambda_1-\lambda_2|^2)e^{-|\lambda_1-\lambda_2|^2}\right)$$

Large-N continued

- **Corollary 2** Local edge scaling limit for conditional overlap

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} D_{11}\left(e^{i\theta}(\sqrt{N}+\lambda_1\right) = \frac{e^{-\frac{1}{2}(\lambda_1+\lambda_1^*)^2} - \frac{\sqrt{2\pi}}{2}(\lambda_1+\lambda_1^*)\text{erfc}\frac{(\lambda_1+\lambda_1^*)}{\sqrt{2}}}{\sqrt{2\pi^3}}$$

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  When all eigenvalues in the bulk have a large large separation: $\lambda_i - \lambda_j \geq \forall i \neq j = 1, \ldots, k$ we have

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- the proof uses a relation between the overlap and eigenvalues correlation functions

Limiting relation in the bulk

$$D_{11}^{\text{bulk}}(\lambda_1, \ldots, \lambda_k) =$$
$$\prod_{l=2}^{k} \left(\frac{-1 - |\lambda_1 - \lambda_l|^2}{|\lambda_1 - \lambda_l|^4} \right) \left(1 - |\lambda_1 - \lambda_l|^2 - (\lambda_1 - \lambda_l) \frac{\partial}{\partial \lambda_l} \right)$$
$$\rho^{\text{bulk}}(\lambda_1, \ldots, \lambda_k)$$

- this hints at the possibility that the known universality of complex eigenvalue correlation functions could be transferred to the overlaps

Summary . . .

- diagonal and off-diagonal overlapp are part of a DPP
- corresponding kernels computed explicitly for finite-N
- local large-N limits in bulk (origin) and at edge follow

. . . and open questions

- further explicit examples for which the DPP remains intact
- the computation of the kernel at finite-N is difficult to generalise
- are the local eigenvector correlations universal?
- global correlations from finite-N results?

ZiF Zentrum für interdisziplinäre Forschung
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RANDOMNESS IN PHYSICS AND MATHEMATICS

FROM STOCHASTIC PROCESSES TO NETWORKS

12 – 24 AUGUST 2019

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