Universality of random matrix dynamics

joint work with Gernot Akemann and Mario Kieburg

arxiv:1809.05905

Random Matrix Theory
Application in the Information Era
April 29th - May 3rd, Kraków

AGH

Z.Burda

Outline

- statics/dynamics of random matrix
- additive, multiplicative stochastic processes
- products of Ginibre matrices
- evolution of local statistics
- discrete time ⟷ depth of the system
- phase transition between deep/shallow systems
- WSR = width-to-spacing ratio
- kernel (WSR)
- conjecture: universality(symmetry + WSR)

Complex Ginibre matrix

$$G = [G_{ij}]_{i=1,\ldots,N,j=1,\ldots,N}$$

$$G_{ij} \sim \text{iid complex } \mathcal{N}(0,1)$$

Rescaled version

$$g = G/\sqrt{N} = \left[G_{ij}/\sqrt{N}\right]_{i=1,\ldots,N,j=1,\ldots,N}$$

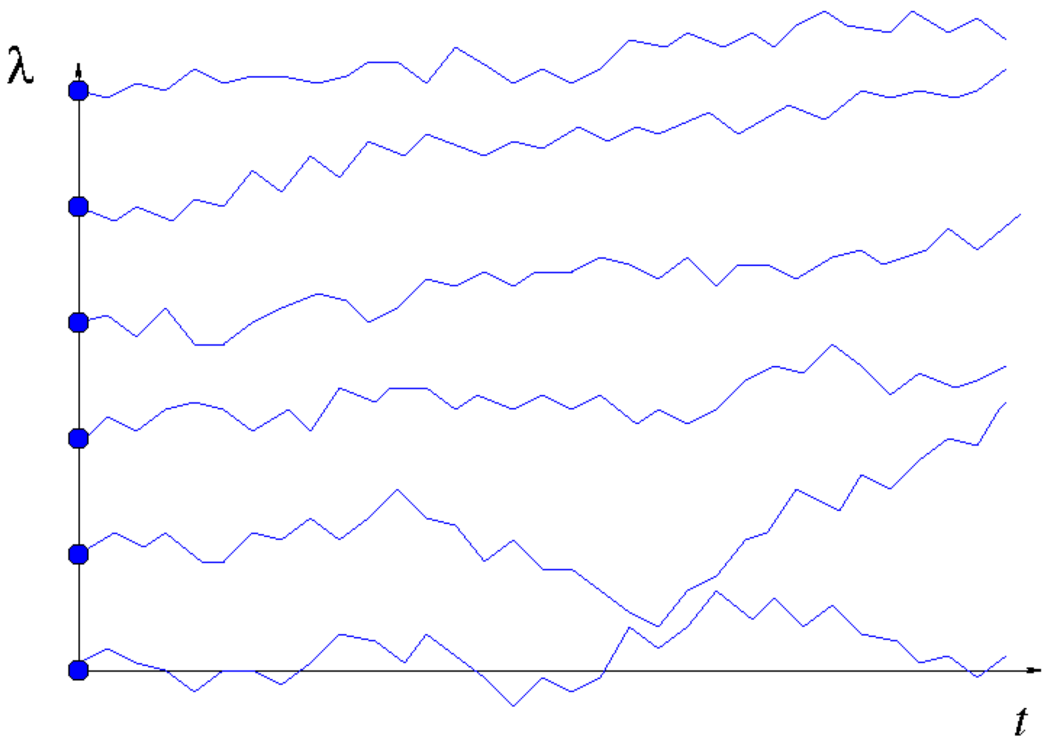
- Dyson Random Walk

$$X_M = \sigma G_M + X_{M-1}$$

$$X_M = \sigma(G_M + G_{M-1} + \ldots + G_1) + X_0$$

- Eigenvalues of $H = \frac{1}{2}\left(X_M + X_M^\dagger\right)$

$$\lambda_1, \lambda_2, \ldots, \lambda_N$$



- Continuum limit

$$M, N \to \infty \ ; \quad t = M\Delta t \ ; \quad \sigma \sim \sqrt{\Delta t} \ ; \quad \Delta t \to 0$$

- Multiplicative stochastic process

$$X_M = G_M X_{M-1}$$

$$X_M = G_M G_{M-1} \ldots G_1 X_0 \ , \quad X_0 = 1$$

$$|x\rangle_M = G_M G_{M-1} \ldots G_1 |x\rangle_0$$

- Eigenvalues of $Y = (G_M \ldots G_1)^\dagger (G_M \ldots G_1)$

- Lyapunov exponents

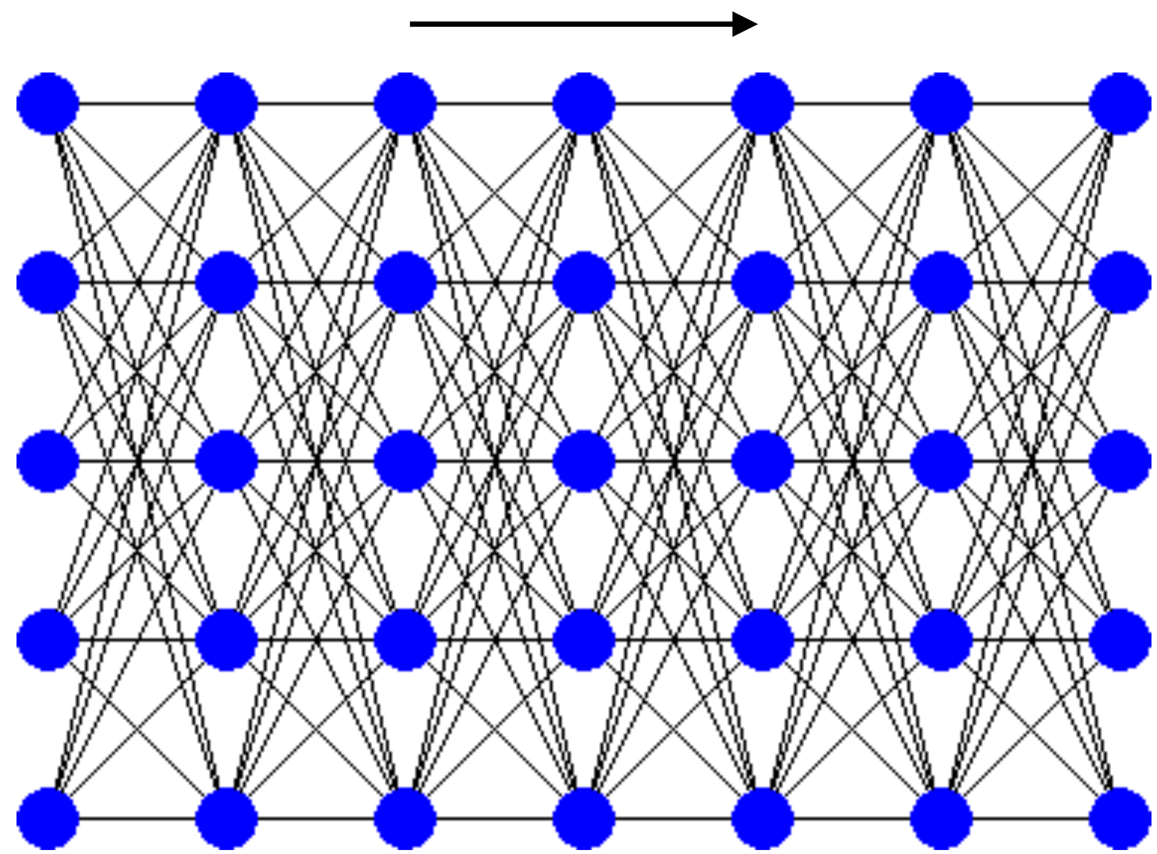
$$L = \frac{1}{2M} \log Y \ , \quad \mu_j = \frac{1}{2M} \log \lambda_j \ , \quad j = 1, \ldots, N$$

- Limit $M, N \to \infty$

Multilayered systems

$$M = 7 \quad N = 5$$

$|x\rangle_0$



$|x\rangle_M$

- Deep systems $M \gg N$
- Critical systems $M \sim N$
- Shallow systems $M \ll N$

Limit $N \to \infty$

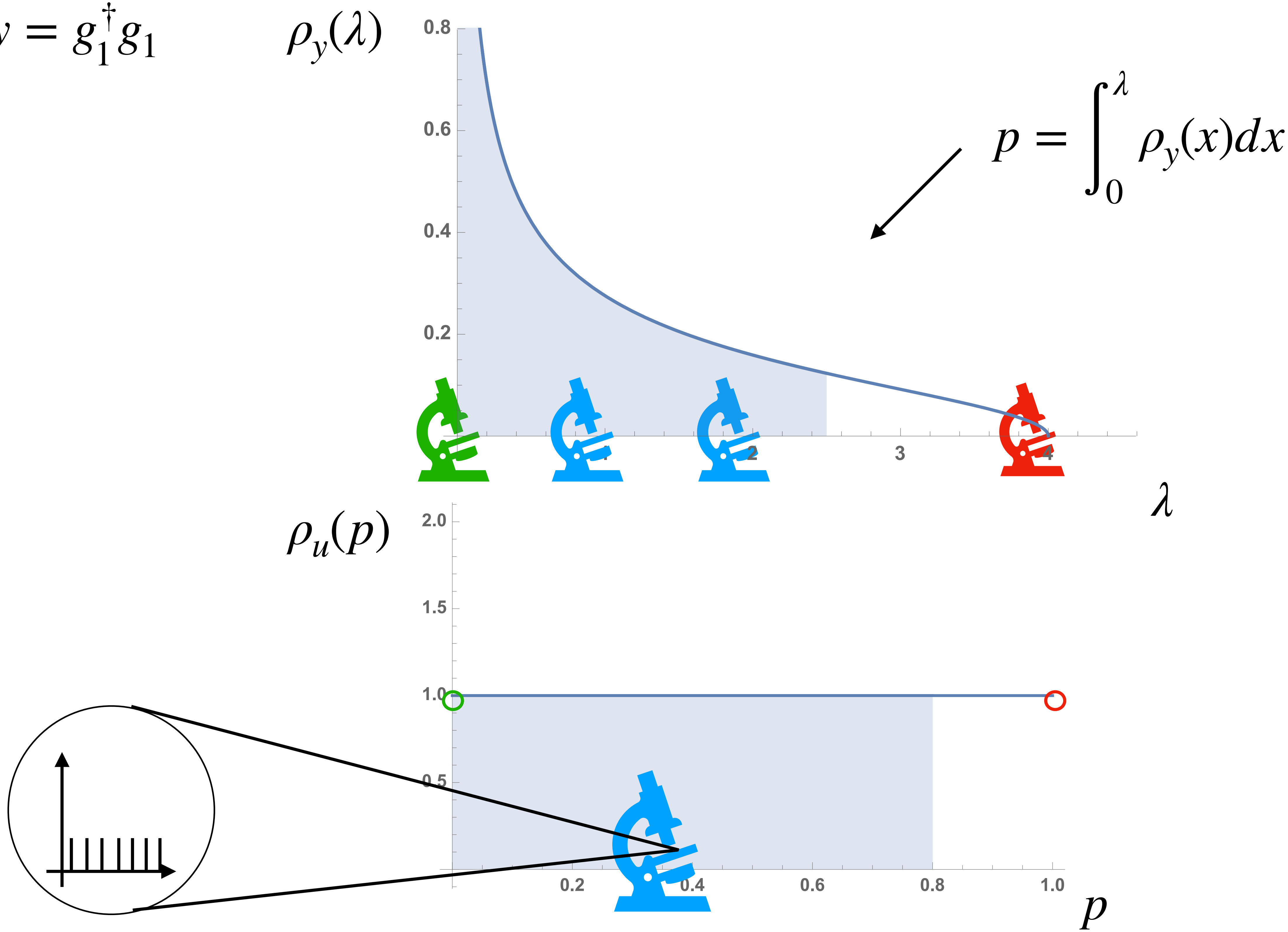
$M = M(N)$ increasing function

$$a = \lim_{N \to \infty} \frac{N}{M(N)}$$

- Deep systems $a = 0$ e.g. $M \sim N^2$
- Critical systems $0 < a < \infty$ e.g. $M \sim N$
- Shallow systems $a = \infty$ e.g. $M \sim N^{1/2}$

Macroscopic density / microscopic density

$y = g_1^\dagger g_1$



Macroscopic density of $y = (g_M \cdots g_1)^\dagger (g_M \cdots g_1)$ for $N \to \infty$

T. Neuschel 2014

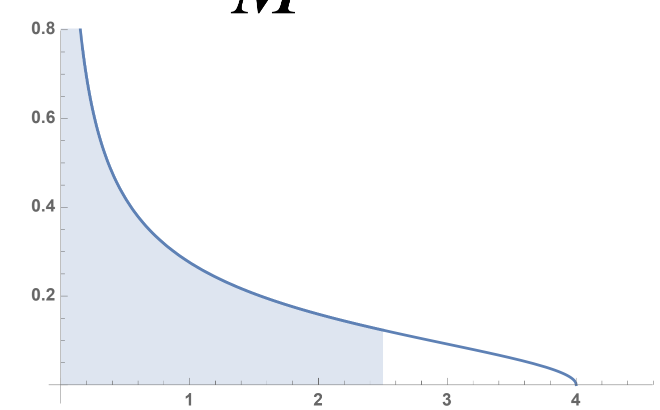
$$x(\phi) = \frac{1}{\sin(\phi)} \frac{\left(\sin\left((M+1)\phi\right)\right)^{M+1}}{\left(\sin(M\phi)\right)^{M}}, \quad \phi \in \left(0, \frac{\pi}{M+1}\right)$$

$$\rho_y(\phi) = \frac{1}{\pi} \frac{\left(\sin(\phi)\right)^2 \left(\sin(M\phi)\right)^{M-1}}{\left(\sin\left((M+1)\phi\right)\right)^{M}}$$

Support: $x \in (0, x_*)$ where $x_* = \frac{(M+1)^{(M+1)}}{M^M}$

Hard edge $\rho_y(x) \sim x^{-M/(M+1)}$

Soft edge $\rho_y(x) \sim (x_* - x)^{1/2}$



Limiting density for $N \to \infty, M \to \infty$

$$u = y^{1/M} = \left((g_M \cdots g_1)^\dagger (g_M \cdots g_1)\right)^{1/M}$$

Eigenvalues of *u* are uniformly distributed on (0,1)

Unfolding $\quad y \to u = y^{1/M}\ ; \quad y = u^M \quad \text{or} \quad Y = (Nu)^M$

Zooming in $\quad \lambda = (Np + \zeta)^M$

Exact result for finite M,N

$$R_{Y,k}(\lambda_1, \ldots, \lambda_k) = \det\left[K_Y(\lambda_i, \lambda_j)\right]_{i,j=1,\ldots,k}$$

$$K_Y(x, y) = \frac{1}{x} \sum_{j=1}^{N} \left(\frac{x}{y}\right)^j G_j(y),$$

$$G_j(y) = \int_{-i\infty}^{+i\infty} \frac{dt}{2\pi i} \frac{\sin(\pi t)}{\pi t}\, y^t \left(\frac{\Gamma(j-t)}{\Gamma(j)}\right)^{M+1} \frac{\Gamma(N-j+1+t)}{\Gamma(N-j+1)}$$

G. Akemann, Z. Burda, 2012

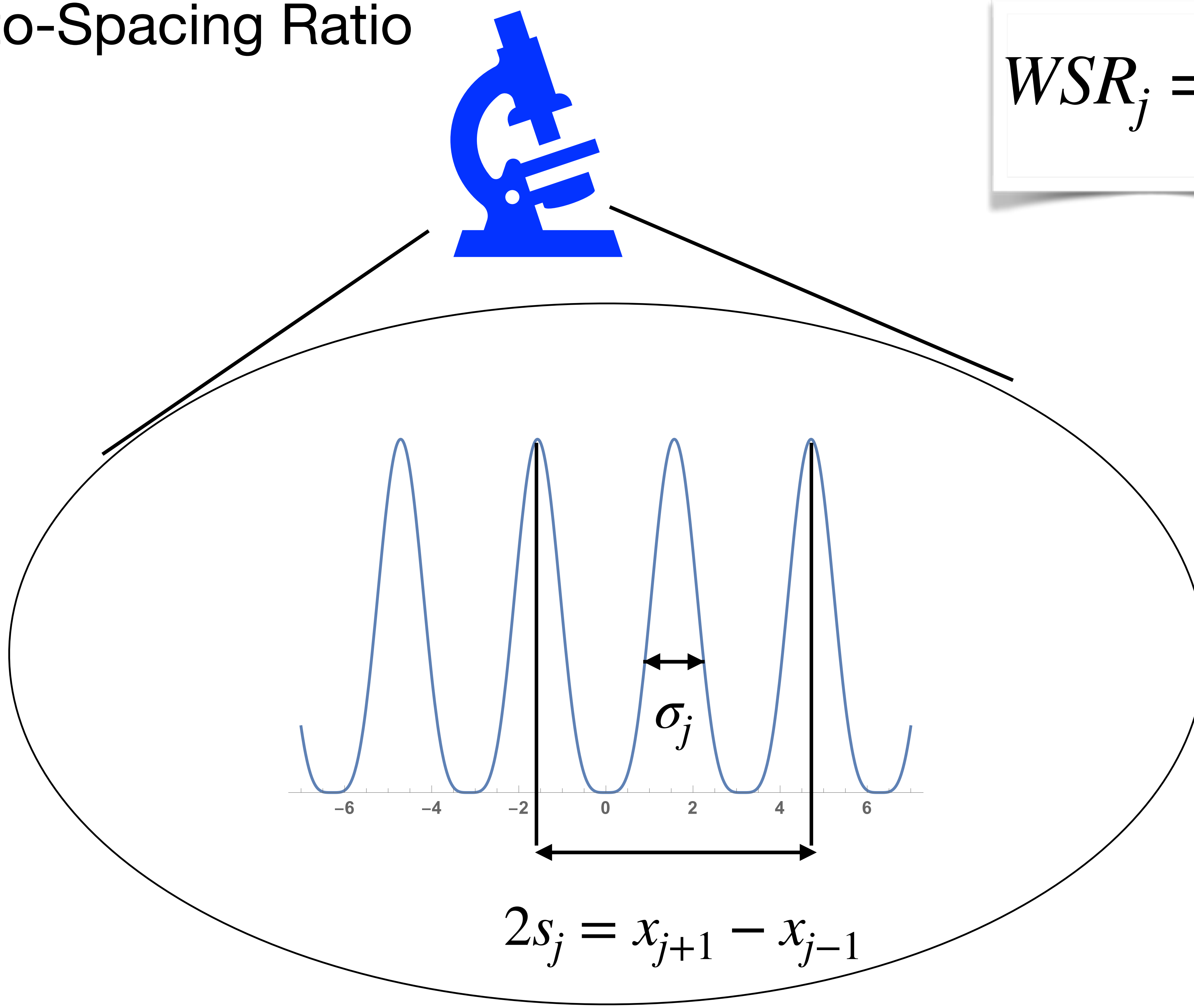
G. Akemann, M. Kieburg, L. Wei, 2013

G. Akemann, Z. Burda, M. Kieburg, 2014

D.-Z. Liu, D. Wang, and L. Zhang, 2014

Width-to-Spacing Ratio

$$WSR_j = \frac{\sigma_j}{s_j}$$



$WSR_j \ll 1$ discrete spectrum

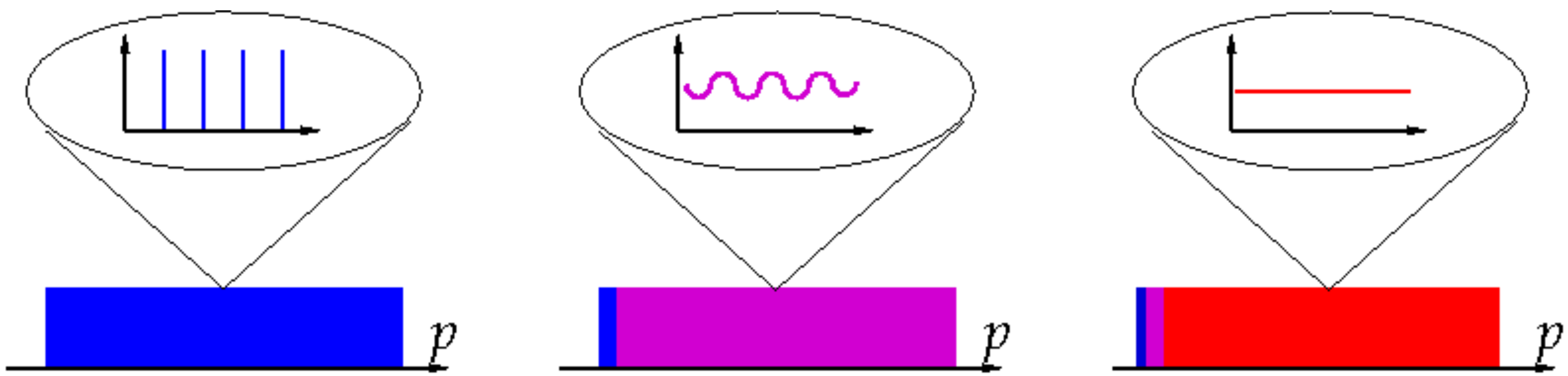
$WSR_j > 1$ contiuous spectrum

For the product of *M* Ginibre matrices of dimension *N x N*

$$WSR_j \approx \sqrt{\frac{j}{M}} \quad \text{for} \quad j = 1,\ldots,N$$

$$WSR_N \approx \sqrt{\frac{N}{M}} \quad \text{for the rightmost peak}$$

In the limit $N \to \infty$, $N/M \to a$



$a = 0$ $\quad$ $0 < a < \infty$ $\quad$ $a = \infty$

$$WSR_{j=pN} \approx \sqrt{ap}$$

Erfi kernel

$$K_{\text{bulk}}(\xi,\zeta) = \frac{1}{2\pi ap}\text{Re}\sum_{\nu=-\infty}^{+\infty}\exp\left(\frac{\nu(\xi-\zeta)}{ap}\right)\text{erfi}\left(\frac{\pi\sqrt{2ap}}{2}+i\frac{\zeta-\nu}{\sqrt{2ap}}\right)$$

G. Akemann, Z. Burda, M. Kieburg, arxiv:1809.05905

D.-Z. Liu, D. Wang, Y. Wang, arxiv:1810.00433

Deep systems $(a = 0)$

$$K_{\text{bulk}}(\xi,\zeta) = \sum_{\nu=-\infty}^{+\infty}\delta(\xi-\nu)$$

Shallow systems $(a = \infty)$

$$K_{\text{bulk}}(\xi,\zeta) = \frac{\sin(\pi(\xi-\zeta))}{\pi(\xi-\zeta)}$$

Local level statistics in the bulk

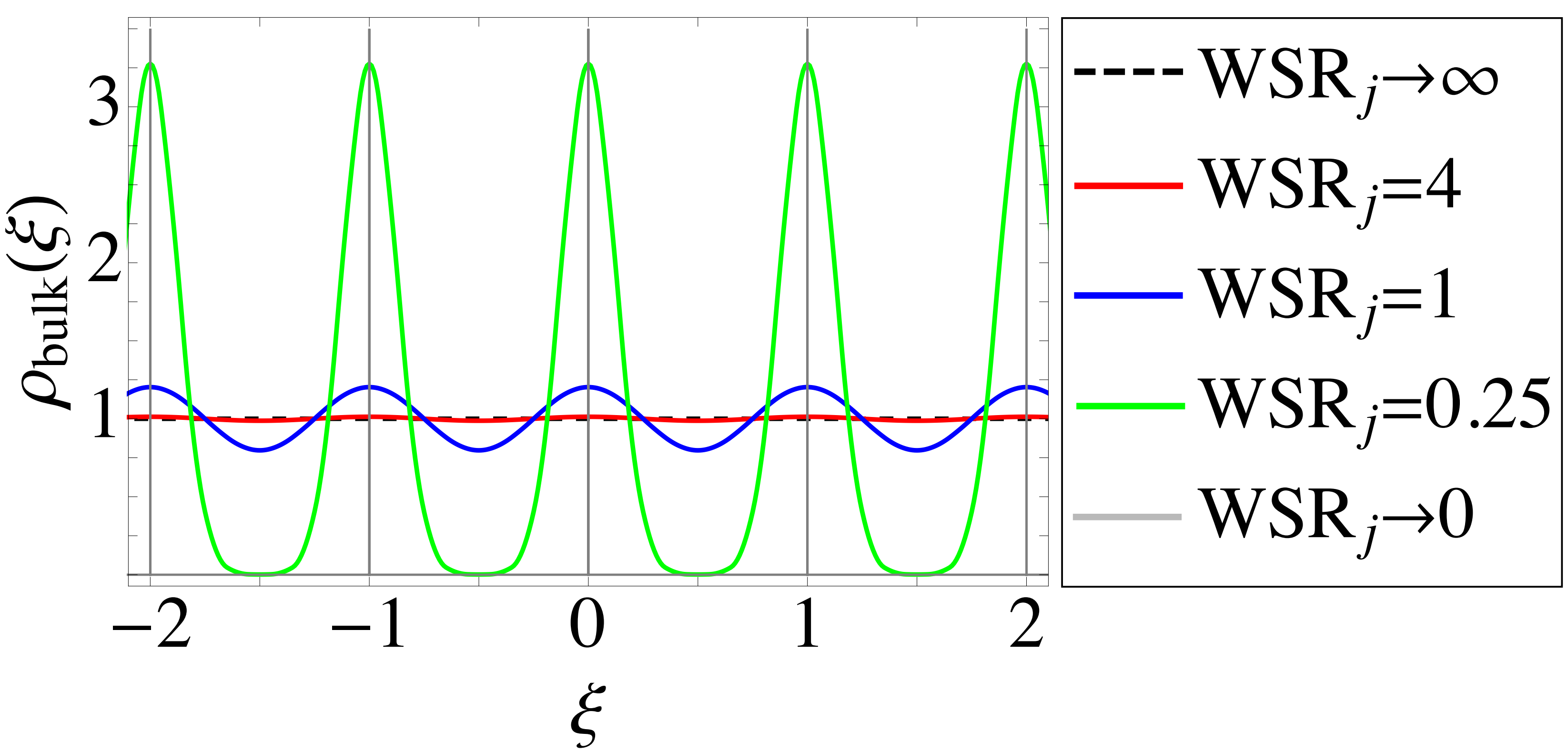
$$\rho_{\mathrm{bulk}}(\xi) = \frac{1}{2\pi a p} \mathrm{Re} \sum_{\nu=-\infty}^{+\infty} \mathrm{erfi} \left(\frac{\pi\sqrt{2ap}}{2} + i \frac{\zeta - \nu}{\sqrt{2ap}} \right)$$

Deep systems $(a = 0)$

$$\rho_{\mathrm{bulk}}(\xi) = \sum_{\nu=-\infty}^{+\infty} \delta(\xi - \nu)$$

Shallow systems $(a = \infty)$

$$\rho_{\mathrm{bulk}}(\xi) = 1$$



Soft edge

Explicit integral representation of the kernel (a)

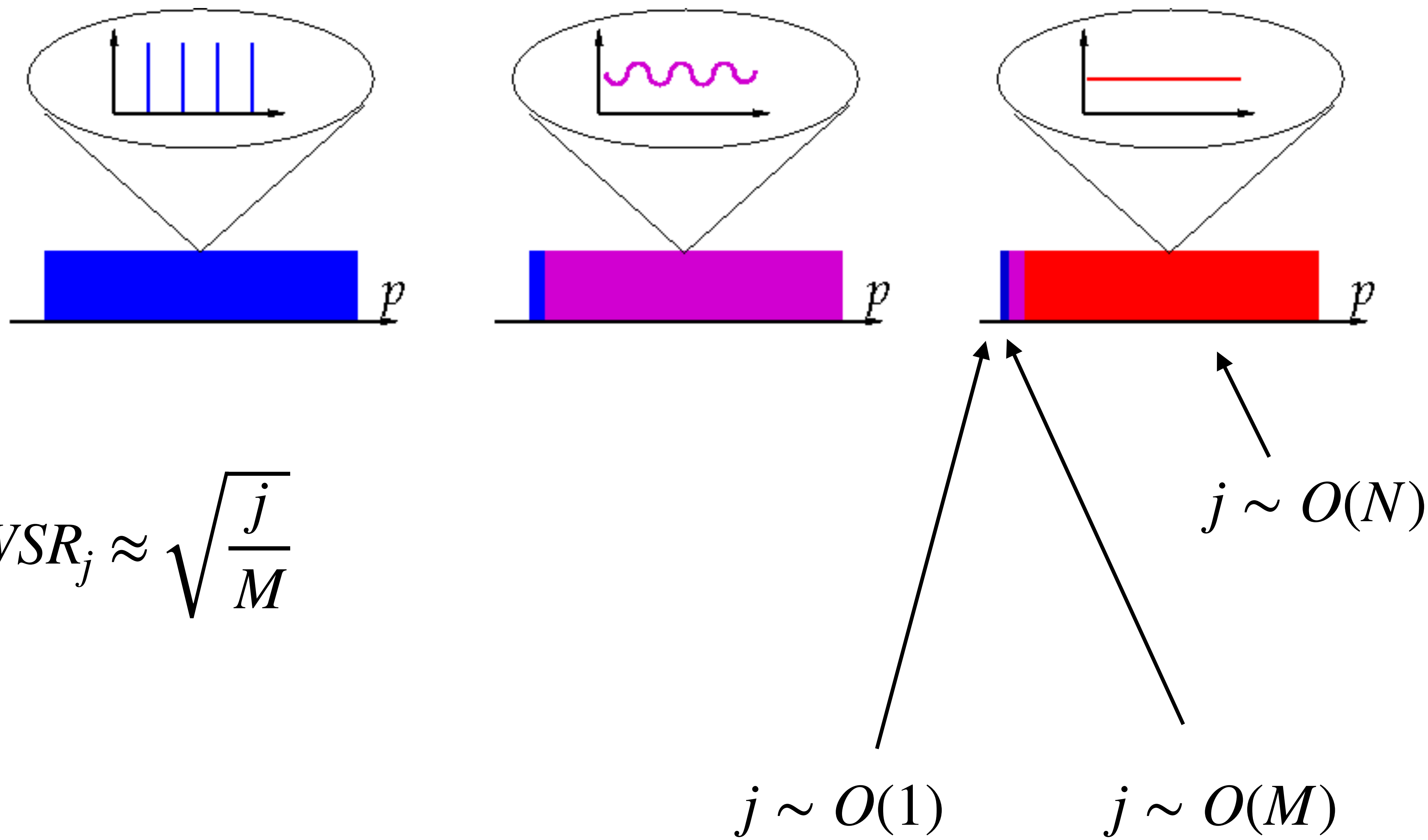
Dirac delta statistics $(a = 0)$ $\longleftrightarrow$ Airy statistics $(a = \infty)$

Hard edge

Dirac delta statistics $(a = 0)$

$$\rho_{\mathrm{LE}}(\mu) = \sum_{j=1} \delta \left(\mu - \frac{\psi(j)}{2} \right)$$

Transitional statistics



$$WSR_j \approx \sqrt{\frac{j}{M}}$$

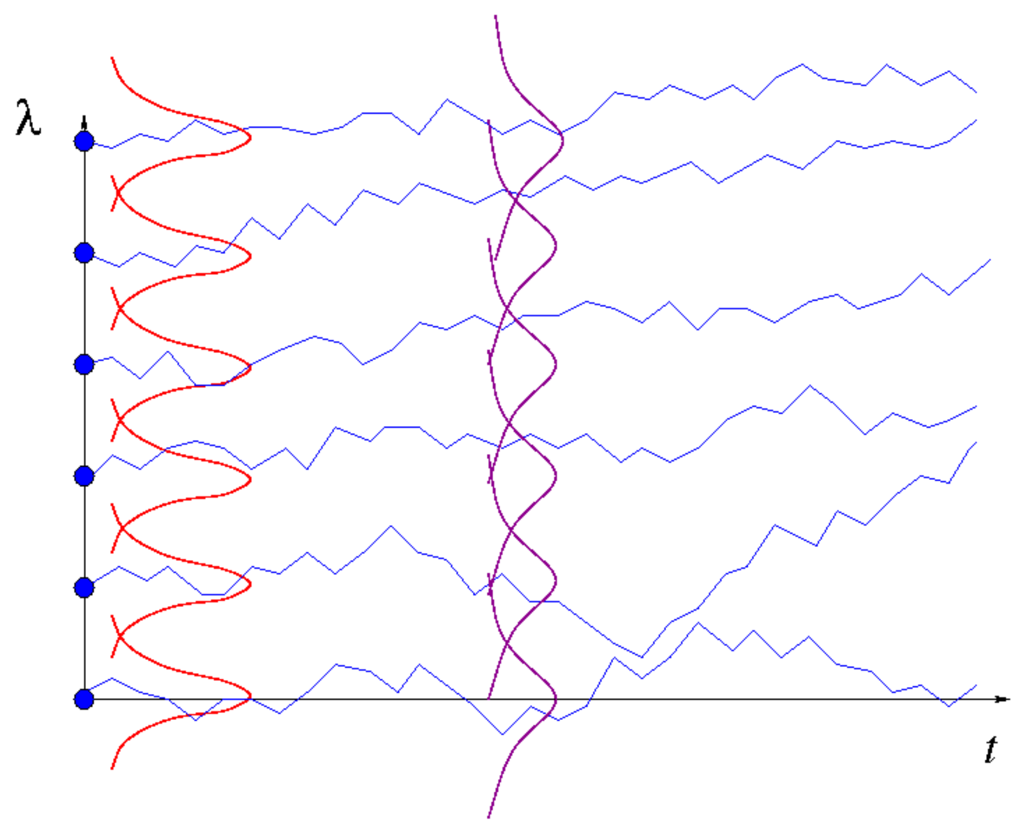
Dysonian random walk

K. Johansson, arXiv:math/0404133

M. Duits, J. Verbaarshot

$$\rho(x, t=0) = \sum_{j=-\infty}^{+\infty} \delta(x - \alpha j) \longrightarrow \rho(x, t=\infty) = 1$$

Picket fences

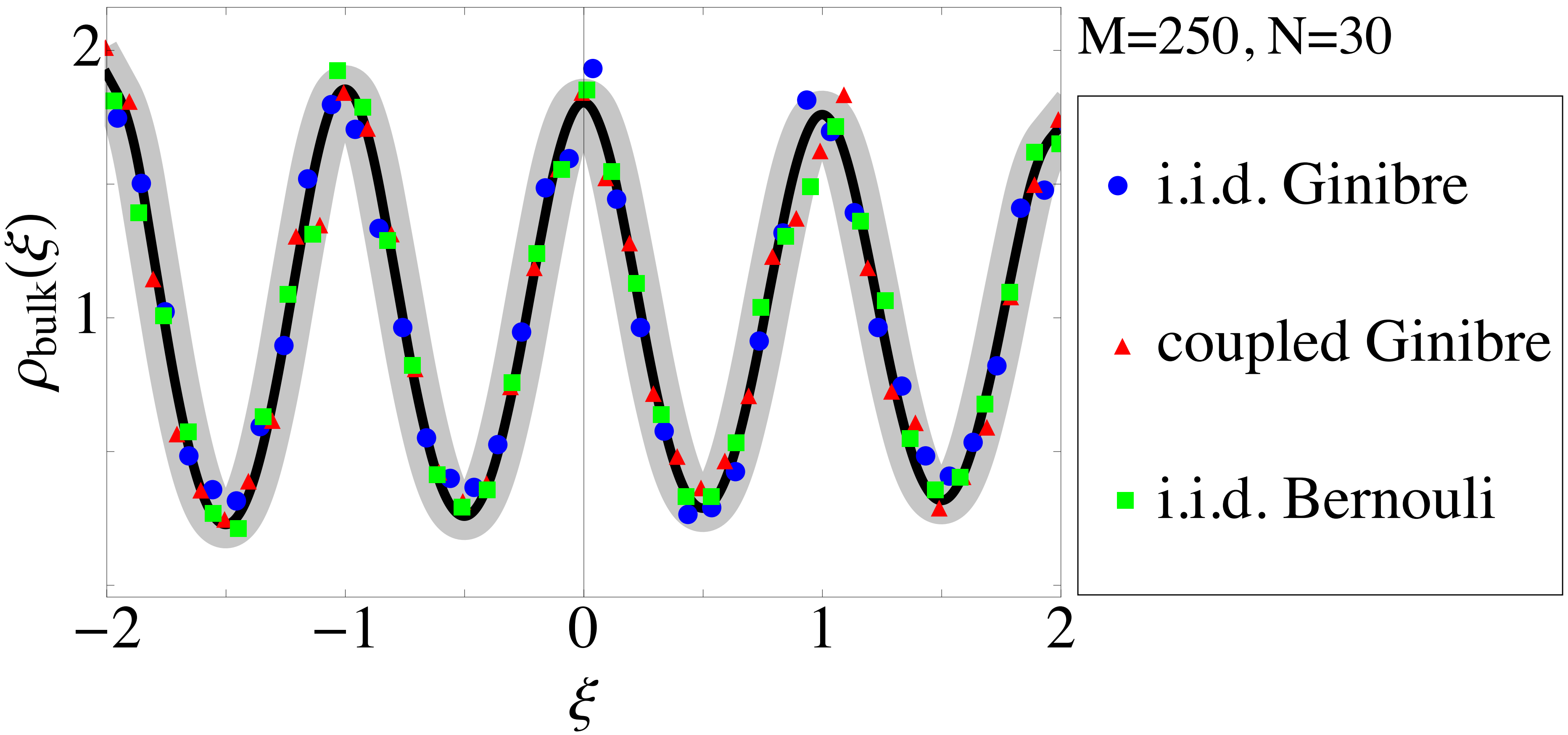


GUE

$$K_{\text{bulk}}(\xi, \zeta) = \frac{1}{\pi}\mathrm{Re} \sum_{k=-\infty}^{+\infty} \exp\left(-2\pi^2 apk(k-1)\right) \frac{\exp\left(i\pi(\zeta + (2k-1)\xi)\right)}{2\pi apk + i(\zeta - \xi)}$$

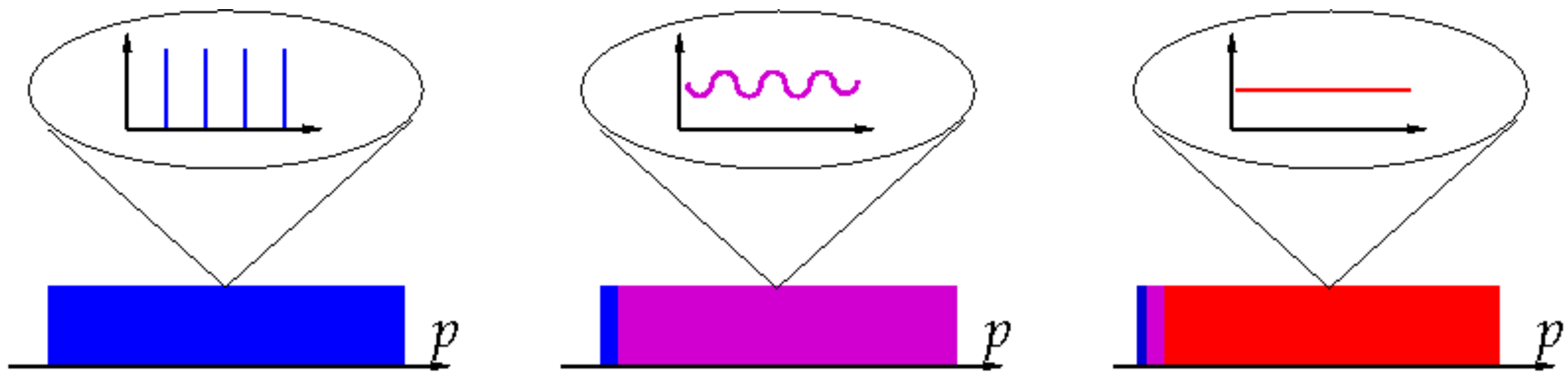
Universality conjecture

Local statistics depends on symmetry class and WSR



Summary

$$Y = (G_M \ldots G_1)^\dagger (G_M \ldots G_1)$$
$$N, M \to \infty \quad , \quad N/M \to a$$



deep $M \gg N$ | critical $M \sim N$ | shallow $M \ll N$

- local statistics depends on WSR (width-to-spacing ratio)
- bulk, soft edge, hard edge, transitional region
- local statistics identical for DRW and product of Ginibre matrices
- universality conjecture: symmetry class + WSR

Thank you!